- Momento angular orbital $\vec{L}$

- Regresamos a $\vec{L}$.
- Describiremos en $\{|\vec{r}\rangle\}$
  - Encontraremos que ev L^2 son $\hbar^2 l(l+1)$ para $l \geq 0$ <u>enteros</u>
- Usando que en $\{|\vec{r}\rangle\}$ $\vec{R}$ es $\vec{r}$ y $\vec{P}$ es $-i\hbar\nabla$

$$\vec{L} = \vec{R} \times \vec{P}$$

$$L_x = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$L_y = \frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

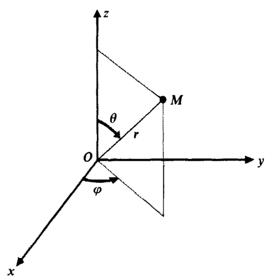
$$L_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

Usando coordenadas esféricas

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

with:

$$\begin{cases} r \geqslant 0 \\ 0 \leqslant \theta \leqslant \pi \\ 0 \leqslant \varphi < 2\pi \end{cases}$$



$$d^3r = r^2 \sin\theta \, dr \, d\theta \, d\varphi$$
$$= r^2 dr \, d\Omega$$

$$d\Omega = \sin\theta \, d\theta \, d\varphi$$

$$L_x = i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}\right)$$

$$L_y = i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \frac{\sin\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}\right)$$

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}$$

which yield:

$$\mathbf{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)$$

$$L_+ = \hbar\, e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

$$L_- = \hbar\, e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

- Para un $|\psi\rangle$ arbitrario las ecs de ev

$$L^2|\psi\rangle = \hbar^2 \ell(\ell+1)\,|\psi\rangle$$
$$L_z|\psi\rangle = \hbar m\,|\psi\rangle$$

toman la forma

$$\langle\vec{r}|\,L^2\,|\psi\rangle = \hbar^2 \ell(\ell+1)\,\langle\vec{r}\,|\psi\rangle = \hbar^2 \ell(\ell+1)\,\psi(\vec{r})$$
$$\langle\vec{r}\,|\,L_z|\psi\rangle = \hbar m\,\psi(\vec{r})$$

$$\Rightarrow \begin{cases} -\left\{\dfrac{\partial^2}{\partial\theta^2} + \dfrac{1}{\tan\theta}\dfrac{\partial}{\partial\theta} + \dfrac{1}{\sin^2\theta}\dfrac{\partial^2}{\partial\varphi^2}\right\}\psi(r,\theta,\varphi) = l\,(l+1)\,\psi(r,\theta,\varphi) & \text{(D-7-a)} \\ -\,i\dfrac{\partial}{\partial\varphi}\,\psi(r,\theta,\varphi) = m\,\psi(r,\theta,\varphi) & \text{(D-7-b)} \end{cases}$$

- Observaciones:

  - Resultados de sección anterior aplican: para ℓ fijo $m = -\ell, \dots, \ell$

  - r no aparece; sólo importan θ y φ.

    ↳ Podemos usar separación de variables $\psi(r,\theta,\varphi) = f(r)\,Y_\ell^m(\theta,\varphi)$

  - Denotamos a las e.f. $Y_\ell^m(\theta,\varphi)$

$$L^2\,Y_\ell^m = \ell(\ell+1)\hbar^2\,Y_\ell^m$$
$$L_z\,Y_\ell^m = \hbar m\,Y_\ell^m$$

— Valores para ℓ y m

separación de variables $Y_\ell^m = F(\theta)\, G(\varphi)$

$$\frac{\hbar}{i}\frac{\partial}{\partial\varphi} Y_\ell^m = m\hbar\, Y_\ell^m \Rightarrow Y_\ell^m = F_\ell^m(\theta)\, e^{im\varphi}$$

Como $Y_\ell^m(\theta,\varphi) = Y_\ell^m(\theta,\varphi+2\pi)$

$$e^{im\varphi} = e^{im(\varphi+2\pi)} \Rightarrow e^{2im\pi} = 1$$

$\Rightarrow$ m es entero

$\Rightarrow$ ℓ es entero.

Sabemos de la teoría general que

$$L_+ Y_\ell^{m=\ell} = 0$$

$$\left(\frac{d}{d\theta} - \ell\cot\theta\right) F_\ell^\ell(\theta) = 0 \Rightarrow \frac{dF_\ell^\ell(\theta)}{d\theta} = \ell\frac{\cos\theta}{\sin\theta} F_\ell^\ell(\theta)$$

$$\Rightarrow \frac{1}{F_\ell^\ell}\frac{d}{d\theta}F = \ell\frac{\cos\theta}{\sin\theta} \quad \text{es} \quad \frac{f'(\theta)}{f(\theta)}$$

$$\Rightarrow \quad \ln F_\ell^\ell(\theta) = \ell \ln(\sin\theta)$$

$$\Rightarrow \quad F_\ell^\ell(\theta) = \sin^\ell(\theta)$$

$$\therefore Y_\ell^\ell(\theta,\varphi) = N \sin^\ell(\theta)\, e^{i\ell\varphi}$$

Aplicando L_- obtenemos las otras

$$\int_0^{2\pi}\int_0^{\pi} |Y_\ell^\ell(\theta,\varphi)|^2 \sin\theta\, d\theta\, d\varphi = 1$$

$l=0$	$l=1$	$l=2$
$Y_0^0 = \sqrt{\frac{1}{4\pi}}$	$Y_1^{\pm 1} = \pm\sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\varphi}$ $Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta$	$Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta e^{\pm 2i\varphi}$ $Y_2^{\pm 1} = \mp\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta e^{\pm i\varphi}$ $Y_2^0 = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$

La expresión general para cualquier valor de l y m está dada por

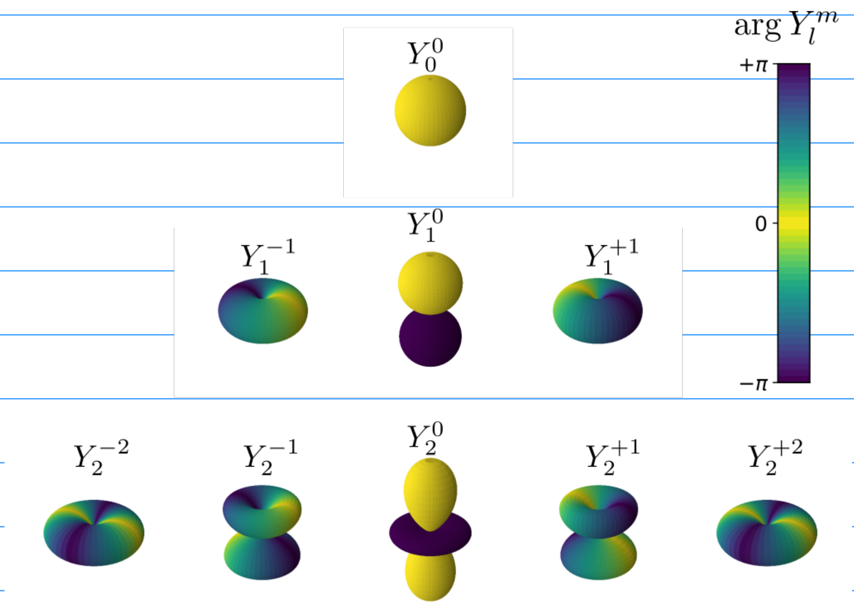
$$Y_l^m(\theta,\varphi) = (-1)^m \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\varphi}, \quad m \geq 0;$$

$$Y_l^{-m}(\theta,\varphi) = (-1)^m Y_l^m(\theta,\varphi)^*;$$

$$P_l^m(x) = (1-x^2)^{|m/2|} \frac{d^{|m|}}{dx^{|m|}} P_l(x);$$

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l}(x^2-1)^l.$$

Aquí, a $P_l^m(x)$ se le conoce como polinomio asociado de Legendre.



Los resultados de L_+, L_-, L^2, L_z aplican para Y_l^m